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On a decomposition of multivariable forms via LMI methods

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Abstract

In this paper, it is shown that some of the convenient characteristics of LMI-based methods can be extended to a class of nonlinear systems. The main idea is to use a computationally tractable sufficient condition for positivity of a function, namely the existence of a "sum of squares" representation. By using an extended set of variables and redundant constraints, it is shown that the conditions can be written as linear matrix inequalities in the unknown parameters. To illustrate the method, we present an example dealing the with Lyapunov stability of systems described by polynomial vector fields.

1 Introduction

In the last few years, Linear Matrix Inequalities (LMI) based methods have demostrated an amazing versatility in the systems and control area [3]. The numerous applications of semidefinite programming to basic applied mathematics problems (mainly, powerful relaxations of NP-hard combinatorial optimization problems, see for example [7]) shows that this trend is bound to continue in the next few years. LMI methods are a lot more than a temporary "control fad," having deep and interesting connections with many important results in other research fields.

Some common, equivalent interpretations of semidefinite programming are to view it as an method to check the positive definiteness of a matrix depending linearly on some parameters, or the nonempty intersection of a linear subspace with the positive semidefinite cone. An alternative viewpoint can also be taken, by considering the LMI conditions as a procedure for checking the existence of a positive definite element in a linear family of *quadratic forms.* As shown in the paper, this interpretation can be extended to more general forms (not necessarily quadratic). By considering a suitable sufficient condition (the existence of a sum of squares decomposition), efficient computational tests can be developed.

As an motivating example of the methodology, we will deal in this paper mainly with the stability analysis of systems described by polynomial vector fields. The same techniques can be employed to many, more complicated problems. Some concrete applications currently being explored are related to parameter-dependent Lyapunov functions, improved conditions for checking the copositivity of a matrix, and the formulation of enhanced semidefinite relaxations for quadratic programming problems.

The paper is organized as follows: in Section 2 some background material on multivariable forms and polynomials is presented. In Section 3 a sufficient condition for positivity is introduced and analyzed, as well as the corresponding algorithm for verifying its applicability. In the next section the computational complexity of the procedure is analyzed. Finally, in the last two sections, an example of the methodology is presented and some conclusions are made.

2 Positivity of forms

Stability analysis can be reduced, using Lyapunov theory, to the existence of a positive definite function, such that its time derivative along the trajectories of the system is negative. As is well known, to prove asymptotic stability of a fixed point of vector field (the origin, without loss of generality) it is required to find a Lyapunov function $V(x)$ such that:

$$\begin{aligned} \dot{x} &= f(x), \qquad V(x) > 0 \quad x \neq 0, \\ \dot{V}(x) &= \left(\frac{\partial V}{\partial x}\right)^T f(x) < 0, \quad x \neq 0 \end{aligned} \tag{1}$$

for all x in a neighborhood of the origin. If we want global results, we need additional conditions such as V being radially unbounded.

In the specific case of linear systems $\dot{x} = Ax$ and quadratic Lyapunov functions $V(x) = x^T P x$ this stability test is equivalent to the well-known LMIs

$$A^T P + PA < 0, \qquad P > 0.$$

The existence of a P satisfying this last condition can be checked efficiently, using for instance interior point methods.

In the attempt to extend this formulation to more general vector fields (not necessarily linear) or Lyapunov functions (not necessarily quadratic), we are faced with the basic question of how to verify in an algorithmic fashion the conditions (1). If we want to develop an algorithmic approach to nonlinear system analysis, similar to what is available in the linear case, we need some explicit way of *testing the global positivity of a function.*

This is a very important problem, and lots of research efforts have been devoted to it. In the specific case of polynomial functions, there is a variety of approaches, see [1] for a survey of existing techniques. An obvious necessary condition is that the degree of the form (or polynomial) be even. It is possible to show that the general problem of testing global positivity of a polynomial function is in fact NP-hard. Therefore, (unless P=NP) every method guaranteed to obtain the right answer in every possible instance will have unacceptable behavior for a problem with a large number of variables. This is the main drawback of theoretically powerful methodologies such as quantifier elimination [5, 10].

If we want to avoid the inherent complexity problems related with the exact solution, the question arises: are there any conditions, that can be tested in polynomial time, to guarantee global positivity of a function? As we will shortly see, one such condition is given by the existence of a sum of squares decomposition.

Before proceeding on to the next section, a brief review of the notation is in order. We will consider in this paper the set $F_{n,m}$ of homogeneous forms of degree m in n variables $\{x_1, \ldots, x_n\}$, with real coefficients. Every such form can be written as a sum of $\binom{n+m-1}{m}$ monomials, each one of the form $c_\alpha \prod_{i=1}^n x_i^{\alpha_i}$, with $\sum_{i=1}^n \alpha_i = m$. If we are dealing with polynomials instead of forms, it is possible to convert ones into the others by homogenization and dehomogenization. Concretely, given any form in $F_{n,m}$, we can obtain a polynomial in $n-1$ variables and degree less than or equal to m by setting any variable (for example, x_n) to the value 1. Conversely, given a polynomial $p(x_1, \ldots, x_n)$ of degree d_1, it can be homogenized into a form of degree $d_2 \geq d_1$ by defining $f(x_1, \ldots, x_n, x_{n+1}) := x_{n+1}^{d_2} p(x_1/x_{n+1}, \ldots, x_n/x_{n+1})$. Homogenization of a polynomial into a form of even degree preserves the property of being positive semidefinite (see [13] for details).

3 A sufficient condition for positivity

A deceptively simple sufficient condition for a real-valued function $F(x)$ to be nonnegative everywhere is given by the existence of a sum of squares decomposition:

$$F(x) = \sum_i f_i^2(x)$$

It is clear that if a given function $F(x)$ can be written as above, for some f_i, then it is nonnegative for all values of x.

However, the question immediately arises: when is such decomposition possible? Naturally, in order for the problem to make sense, some restriction on the class of functions f_i has to be imposed. Otherwise, we can always define f_1 to be the square root of F, making the condition both useless and trivial.

For the case of polynomials, this is a well-analyzed problem, first studied by David Hilbert more than a century ago. In fact, one of the questions in his famous list of twenty-three unsolved problems presented at the International Congress of Mathematicians at Paris in 1900, deals with the representation of a definite form as a sum of squares of rational functions.

For notational simplicity, we will use the notation *psd* for "positive semidefinite" and *sos* for "sum of squares." Following the notation in references [4, 13], let $P_{n,m}$ be the set of psd forms of degree m in n variables, and $\Sigma_{n,m}$ the set of forms p such that $p = \sum_k h_k^2$, where h_k are forms of degree $m/2$.

Hilbert himself noted that not every psd polynomial (or form) is sos. A simple, more modern counterexample is the Motzkin form (here, for $n = 3$)

$$M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \qquad (2)$$

Positive semidefiniteness can be easily shown using the arithmetic-geometric inequality, and the inexistence of a sos decomposition follows from standard algebraic manipulations (see [13] for details), or the procedure outlined below.

Hilbert gave a complete characterization of when these two classes are equivalent. There are three cases for which the equality holds. The first one, is the case of forms in two variables ($n = 2$), which are equivalent by dehomogenization to polynomials in one variable. This is easy to show using a factorization of the polynomial in linear and quadratic factors. The second one, is the familiar case of quadratic forms (i.e. $m = 2$) where the sum of squares decomposition follows from the eigenvalue/eigenvector factorization. There is also a surprising third case, where $P_{3,4} = \Sigma_{3,4}$, corresponding to quartic forms in three variables.

The sum of squares decomposition is the underlying machinery in Shor's global bound for polynomial functions [15], as is explicitly mentioned in [14]. It has also been presented as the "Gram matrix" method in [4] and more recently in [11], although no mention to interior point methods is made: the resulting LMIs are solved via decision methods. A related scheme also appears in [8] (note also the important correction in [6]).

The basic idea of the method is the following: express the given polynomial as a quadratic form in some new variables z. These new variables are the original x ones, plus all the monomials of degree less than or equal to $m/2$ given by the different products of the x variables. Therefore, $F(x)$ can be represented as:

$$F(x) = z^T Q z \tag{3}$$

where Q is a constant matrix. If in the representation above Q is positive semidefinite, then $F(x)$ is also psd. This is the idea in [2], for example, and it can be shown to be conservative, generally speaking. The main reason is that since the variables z_i *are not independent* the representation (3) *might not be unique*, and Q may be psd for some representations, but not for others. Similar issues appear in the analysis of quasi-LPV systems, see [9]. By using identically satisfied constraints that relate the z_i variables among themselves (of the form $z_i z_j = z_k z_l$ or $z_i^2 = z_k z_l$), it is easily shown that there is a linear subspace of matrices Q that satisfy (3). If the intersection of this subspace with the positive semidefinite matrix cone is nonempty, then the original function F is guaranteed to be sos (and therefore psd). This follows from an eigenvalue decomposition of $Q = T^T D T, d_i \geq 0$, which implies the sos $F(x) = \sum_i d_i (Tz)_i^2$. Conversely, if F can indeed be written as the sum of squares of polynomial, then expanding in monomials will provide the representation (3).

Example 1 *The following example is from [1, Example 2.4], where it is required to find whether or not the quartic polynomial*

$$P(x_1, x_2, x_3) = x_1^4 - (2x_2x_3 + 1)x_1^2 + (x_2^2x_3^2 + 2x_2x_3 + 2),$$

is positive definite.

By constructing the Q matrix as above, and solving the corresponding LMIs, we obtain the sums of squares decomposition:

$$P(x_1, x_2, x_3) = 1 + x_1^2 + (1 - x_1^2 + x_2x_3)^2,$$

that immediately establishes global positivity. Notice that the decomposition actually proves a stronger fact, namely that $P(x_1, x_2, x_3) \geq 1$ for all values of x_i.

The most important properties that distinguish this condition from other approaches are its relative *tractability* and the fact that it can be easily extended to the uncertain case (i.e. when we are looking for a psd F, subject to additional conditions).

It should be noted that, at least in principle, the method has some degree of conservativeness. As explained above, this is because the class of psd polynomials is not equal to the sos ones. It is not clear yet how relevant this gap is in practical terms. After all, almost every time the positivity of a function needs to be established (for example, in backstepping methods), this is usually done by constructing a sos representation, either implicitly or explicitly. In any case, there is a possible workaround, at some computational cost. Artin's positive answer to Hilbert's 17th problem assures for a psd $F(x)$ the existence of a polynomial $G(x)$, such that $F(x)G^2(x)$ can be written as a sum of squares. In particular, Reznick's results [12] show that if F is *positive definite* it is always possible to take $G(x) = (\sum x_i^2)^r$, for sufficiently large r.

Example 2 *Consider the case of the Motzkin form given in equation (2). As mentioned before, it cannot be written as a sum of squares of polynomials. Even though it is only semidefinite (so in principle we cannot apply Reznick's theorem), after solving the LMIs we obtain the decomposition:*

$$\begin{aligned}
&(x^2 + y^2 + z^2)\, M(x, y, z) = \\
&(x^2yz - yz^3)^2 + (xy^2z - xz^3)^2 + (x^2y^2 - z^4)^2 + \\
&+ \frac{1}{4}(xy^3 - x^3y)^2 + \frac{3}{4}(xy^3 + x^3y - 2xyz^2)^2,
\end{aligned}$$

from where nonnegativity is obvious.

4 Computational considerations

The computational cost of the procedure clearly depends on both the degree of the polynomial, and the number of variables. The number of monomials of degree less than or equal to $m/2$ (m is even) is $N_z := \binom{n-1+m/2}{m/2}$. This is the size of the resulting LMI, assuming no simplifications occur (which is not usually the case). The number of constraints (additional variables in the LMIs) can be large, especially when using many variables and high degree polynomials. For a fixed degree, however, that number is always a polynomial expression in n (it is always bounded by N_z^2, for instance). A complete analysis of these quantities has been presented in [4].

A minor inconvenience that might happen, depending on the given polynomial and the constraints employed, is that the resulting LMIs can be feasible, but not strictly feasible (i.e. the matrix Q can be made positive semidefinite, but not positive definite). We will

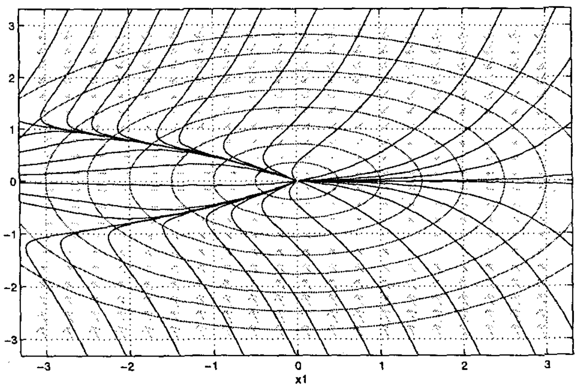

Figure 1: Phase plot and Lyapunov function level sets.

see an instance of this in the example in the next section. In many cases, this can be fixed by a presolving stage, where some variables are eliminated, and the dimension of the problem is reduced. The current LMI solvers (mostly interior-point based) handle these type of problems with varying degree of success. Clearly, more work needs to be done, both at the theoretical and implementation level, in order to deal with these type of problems. Nevertheless, the highly structured semidefinite programs resulting from the proposed approach usually allows for an easily achievable reduction in the dimensionality.

5 Example: stability analysis

In this simple problem, we are looking for a Lyapunov function to prove global stability of a nonlinear system. The system is described by:

$$\begin{aligned}\dot{x}_1 &= -x_1 - 2x_2^2 \\ \dot{x}_2 &= -x_2 - x_1x_2 - 2x_2^3;\end{aligned}$$

Notice that the vector field is invariant under the symmetry transformation $(x_1, x_2) \rightarrow (x_1, -x_2)$. We could potentially use this information in order to limit the search to symmetric candidate Lyapunov functions. However, we will not do so, to show the method in its full generality. To look for a Lyapunov function, we will use the general expression of a polynomial in x_1, x_2 of degree four with no constant or linear terms (because $V(0) = 0$, and V has to be positive definite). We use a matrix representation for notational clarity.

$$V(x) = \begin{bmatrix} 1 \\ x_1 \\ x_1^2 \\ x_1^3 \\ x_1^4 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & c_{02} & c_{03} & c_{04} \\ 0 & c_{11} & c_{12} & c_{13} & 0 \\ c_{20} & c_{21} & c_{22} & 0 & 0 \\ c_{30} & c_{31} & 0 & 0 & 0 \\ c_{40} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_2^3 \\ x_2^4 \end{bmatrix}$$

It is easy to verify that V can be represented as $V(x) = \frac{1}{2}z^TQz$, where $z = [x_1, x_1^2, x_1x_2, x_2^2, x_2]^T$ and

$$Q = \begin{bmatrix} 2c_{20} & c_{30} & c_{21}+\lambda_2 & c_{12}+\lambda_1 & c_{11} \\ c_{30} & 2c_{40} & c_{31} & -\lambda_3 & -\lambda_2 \\ c_{21}+\lambda_2 & c_{31} & 2c_{22}+2\lambda_3 & c_{13} & -\lambda_1 \\ c_{12}+\lambda_1 & -\lambda_3 & c_{13} & 2c_{04} & c_{03} \\ c_{11} & -\lambda_2 & -\lambda_1 & c_{03} & 2c_{02} \end{bmatrix},$$

which λ_i being arbitrary real numbers. The condition for the existence of a sos representation for V, obtained as explained in the paper, is therefore $Q \geq 0$.

For the derivative, we obtain after some algebra that

$$\dot{V}(x) = \begin{bmatrix} 1 \\ x_1 \\ x_1^2 \\ x_1^3 \\ x_1^4 \end{bmatrix}^T A \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_2^3 \\ x_2^4 \\ x_2^5 \\ x_2^6 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & a_{02} & a_{03} & a_{04} & a_{05} & a_{06} \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & 0 \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{30} & a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{40} & a_{41} & a_{42} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

where the a_i are linear functions of the c_i. For example, we have $a_{12} = -4c_{20} - c_{12} - 2c_{12} - 2c_{02}$, and $a_{42} = 0$. The full expressions are omitted for space reasons.

Writing it as a quadratic expression, we have $\dot{V}(x) = -\frac{1}{2}w^TRw$, with the vector $w = [x_1, x_1^2, x_1x_2, x_2^2, x_2, x_1x_2^2, x_1^2x_2, x_2^3]^T$. The expression for the matrix R is given in Table 1.

Again, here ν_i are arbitrary real parameters. If $\dot{V}$ has to be negative, then the *sos* condition is $R \geq 0$. Notice that having $a_{42} = 0$ immediately implies that the multipliers $\nu_9, \nu_{11}, \nu_{12}, \nu_{13}, \nu_{14}$ and the coefficients a_{41}, a_{33} should also vanish. This way, the LMIs are considerably simplified.

After solving the LMIs, it turns out that for this specific example it is even possible to pick a particularly elegant solution, given by a quadratic Lyapunov function. This can be achieved by minimizing the sum of diagonal elements corresponding to the nonquadratic terms, subject to the LMI constraints. In fact, we can choose all the c_i, λ_i and ν_i equal to zero, except $c_{20} = 1$ and $c_{02} = 2$, i.e.

$$V(x_1, x_2) = x_1^2 + 2x_2^2.$$

In this case, we have

$$\begin{aligned}\dot{V}(x) &= (2x_1)(-x_1 - 2x_2^2) + (4x_2)(-x_2 - x_1x_2 - 2x_2^3) \\ &= -4x_2^2 - 2(x_1 + 2x_2^2)^2 \leq 0\end{aligned}$$

In Figure 1 a phase plot of the vector field and the level sets of the obtained Lyapunov function are presented.

$$\begin{bmatrix} 2a_{20} & a_{30} & a_{21}+\nu_6 & a_{12}+\nu_2 & a_{11} & -\nu_7+\nu_8 & -\nu_{13} & a_{13}+\nu_3 \\ a_{30} & 2a_{40} & a_{31}+\nu_{13} & -\nu_8+\nu_9 & -\nu_6 & a_{32}+\nu_{14} & a_{41} & -\nu_{10}+\nu_{11} \\ a_{21}+\nu_6 & a_{31}+\nu_{13} & 2a_{22}+2\nu_7 & -\nu_3+\nu_4 & -\nu_2 & a_{23}+\nu_{10} & -\nu_{14} & a_{14}+\nu_5 \\ a_{12}+\nu_2 & -\nu_8+\nu_9 & -\nu_3+\nu_4 & 2a_{04}+2\nu_1 & a_{03} & -\nu_5 & -\nu_{11} & a_{05} \\ a_{11} & -\nu_6 & -\nu_2 & a_{03} & 2a_{02} & -\nu_4 & -\nu_9 & -\nu_1 \\ -\nu_7+\nu_8 & a_{32}+\nu_{14} & a_{23}+\nu_{10} & -\nu_5 & -\nu_4 & 2a_{24}+2\nu_{12} & a_{33} & a_{15} \\ -\nu_{13} & a_{41} & -\nu_{14} & -\nu_{11} & -\nu_9 & a_{33} & 2a_{42} & -\nu_{12} \\ a_{13}+\nu_3 & -\nu_{10}+\nu_{11} & a_{14}+\nu_5 & a_{05} & -\nu_1 & a_{15} & -\nu_{12} & 2a_{06} \end{bmatrix}$$

Table 1: The matrix R.

It should be remarked that the considerable simplification in the final answer is not really necessary. Any feasible solution of the LMIs will provide a stability-proving Lyapunov function.

6 Conclusions

The sum of squares decomposition is a very useful sufficient condition for positivity of a multivariable polynomial. It can be obtained at a reasonable computational cost, using LMI methods. One of the big advantages of the proposed procedure is that it is a completely algorithmic procedure. All the computations can be carried through in a deterministic fashion, in polynomial time.

The basic idea of the procedure can be applied to numerous problems in the systems and control area. In this paper we have shown one immediate application, dealing with the analysis of global stability of nonlinear systems via polynomial Lyapunov functions. By using the methodology described in this paper, the search for a stability-proving Lyapunov function can be efficiently carried out, generalizing very succesful schemes for the linear case.

Of course, one of the main differences between linear and nonlinear control is that in the latter, *global behavior* is not necessarily the most important consideration: in applications, many successful nonlinear schemes are *not* globally stable. However, tools used successfully in the linear case to deal with these problems (for example, S-procedure) can usually be applied, with suitable modifications.

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